ETEAM

Problems for the first ETEAM

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Foreword

The problems that follow are difficult and are proposed by researchers and students in mathematics. To the best of the knowledge of the authors, they do not always admit a complete solution. However, they are accessible to high school students, i.e. the authors are certain that elementary research work can be carried out on these problems. The jury does not expect the candidates to solve a problem entirely, but rather to understand the issues, solve particular cases, identify difficulties and suggest directions of research. The questions are not always arranged in increasing order of difficulty. Finally, it is not necessary to solve all the problems: each team can reject a certain number of problems without penalty. Please refer to the rules for further details.

Contents

Keywords

1. Geometry 2. Combinatorics 3. Probability 4. Dynamical systems
5. Arithmetic 6. Analysis, Optimization 7. Game theory 8. Combinatorics, Geometry
9. Optimization 10. Game theory, Graphs

Notations

$\mathbb{R}, \mathbb{Z}$	The sets of real numbers and integer numbers, respectively
$[a,b)$	The set of all numbers $x \in \mathbb{R}$ such that $a \leqslant x < b$
$\mathbb{N}$	The set of strictly positive integer numbers $\{1,2,\ldots\}$
$[\![n,m]\!]$	The set of integers $\{n,n+1,\ldots,m\}$

1. Exploring Flatland

In Flatland the reality takes place inside the plane. The objects in Flatland are convex shapes. The Flatlanders have each one punctual eye, which is capable of measuring angles.

At a point P outside of an object $\mathcal{S}$, there are exactly two **tangent rays** from P to $\mathcal{S}$ (the rays through P intersecting the boundary of $\mathcal{S}$ without intersecting its interior). The angle between the two tangent rays is called the **angle of sight** at P. The eye of a Flatlander allows one to determine the angle of sight.

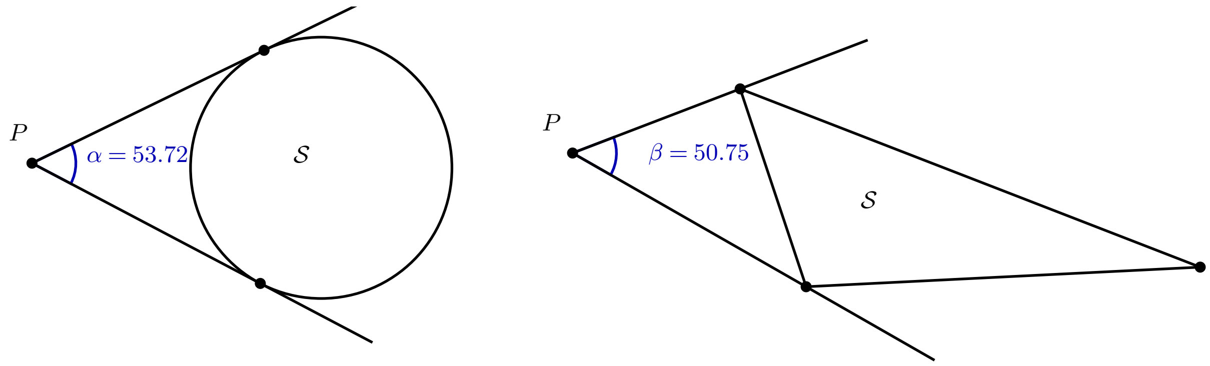

Figure 1. Example of two objects with angle of sight.

1. In the case where $\mathcal{S}$ is a disk with center at the origin and radius 1, what is the angle of sight at $P = (x, y)$ (outside of $\mathcal{S}$)?

Edwin is a Flatlander who dislikes changing sight. Whenever he looks at an object, he goes around it by keeping the same angle of sight.

2. Determine the path Edwin must take to go around a square. Describe the path for any convex polygon.

Emily is a Flatlander and an explorer. She wants to know the exact shape of the objects she encounters. To this end, Emily has a device which allows to determine tangent rays.

3. With a finite number of measurements, is it always possible for Emily to determine the exact shape

a) of an arbitrary object?
b) of an n-gon for some fixed $n \geqslant 3$ (which is known to Emily)? If yes, give a bound for how many measurements she has to do. One can start with the case of a triangle.

One day, Edwin and Emily walk together on a circle C, focusing on a bounded shape $\mathcal{S}$ under a constant angle. Emily forgot to take her device to determine tangent rays. She wonders whether $\mathcal{S}$ has to be a disk.

4. Under which conditions on the angle of sight α, Emily can deduce that the shape is a disk? One can start by treating the following cases:

a) α is an irrational multiple of π.
b) $\alpha = \pi/3$.
c) α is a rational multiple of π.

Naturally Edwin likes shapes where he can walk on a straight line ℓ, while keeping the same angle of sight.

5. For given straight line ℓ and angle of sight α, describe the space of all such shapes $\mathcal{S}$. In particular, consider the following cases:

a) $\mathcal{S}$ is bounded (i.e. of finite size).

b) $\mathcal{S}$ is unbounded. One can start by considering $\alpha < \pi/2, \alpha = \pi/2$ and then $\alpha > \pi/2$. Are there examples where $\mathcal{S}$ has smooth boundary?

6. Suggest and study further directions of research, for example in 3 dimensions.

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2. A MAZING HIVE

A particular species of bees builds its hives as labyrinths. A **hive** of size $N > 0$ is built as follows:

- We consider a regular hexagon whose edges are of length N;
- We pave it regularly by regular triangles with edges of length 1;
- For each vertex V of those triangles we put a cell of the hive which is a regular hexagon with center V and edges of length $1/\sqrt{3}$, all oriented in the same way, so that two cells with respective centers V_1 and V_2 share a common edge if, and only if, there is a regular triangle having $[V_1, V_2]$ as an edge.

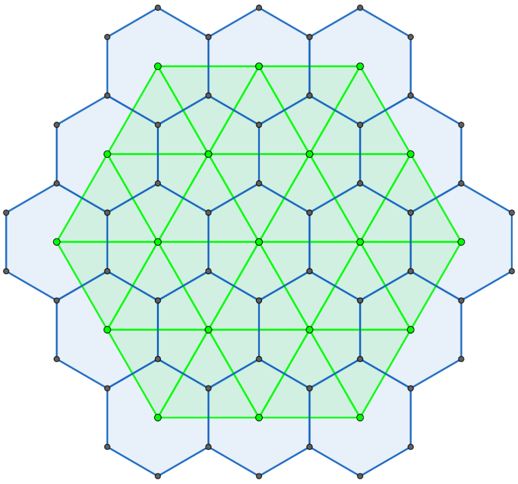

FIGURE 2. Example of a hive, in blue, of size $n = 2$.

To plan their work, the bees label the edges on the perimeter of the hive from 1 to M, starting by an extremal point of its convex hull. In each cell, the bees build exactly 3 walls as follows:

- The walls are line segments between the center of the cell and a vertex of the cell;
- Any 2 walls of a given hexagonal cell form an angle of $120° = (2\pi/3)$rad;
- These walls cut the hexagonal cell in 3 rhombuses.

The datum of all the wall positions is called the **hive configuration**.
We can now see paths inside of a hive. By definition, a **path** is a finite sequence of $k \in \mathbb{N}^\star$ rhombuses $(R_1, \ldots, R_k)$ such that:

- If $i \neq j$, then $R_i \neq R_j$;
- The rhombuses R_i and R_j have a common edge $E_{i,j}$;
- The edge $E_{i,j}$ is not a wall.

The integer k is called the **length** of the path. A path is said to be **maximal** if there does not exist neither R_0 such that $(R_0, R_1, ..., R_k)$ is a path nor R_{k+1} such that $(R_1, ..., R_k, R_{k+1})$ is a path. A path is said to be **crossing** when R_1 and R_k meet the perimeter of the hive.

1. Depending on N, does there exist

a) a crossing path that is not maximal?
b) a maximal path that is not crossing?

2. Depending on N, how many

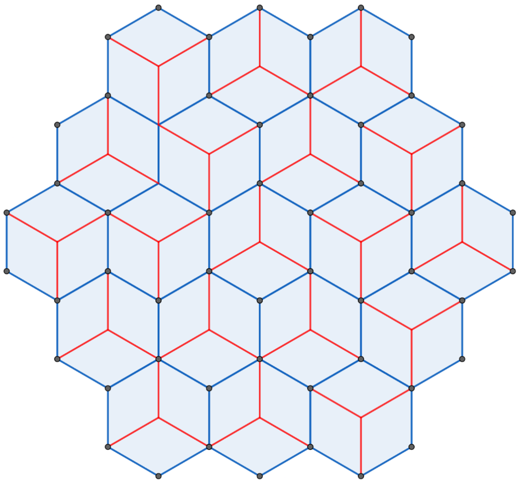

FIGURE 3. Example of a hive with walls printed in red.

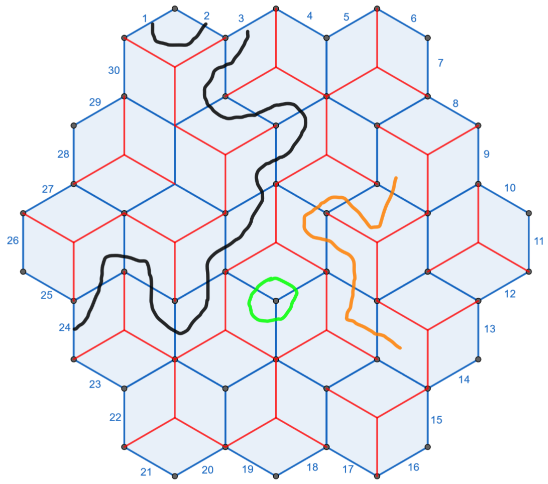

FIGURE 4. The black paths are crossing paths between edges $1 \leftrightarrow 2$ and $3 \leftrightarrow 24$.
The orange path is not a maximal path.
The green path is a maximal non-crossing path.

a) edges M are there on the perimeter?
b) different hives can the bees build, up to a rotation of the hexagon?
c) maximal paths can there be in a given hive?
d) different hives can the bees build, such that all maximal paths are crossing?

3. What are the possible lengths for the longest:

a) crossing path in the hive?
b) path in the hive?

The bees are now biased, each hexagon has a probability p to have a vertical wall pointing to the top and a probability $1-p$ of having a vertical wall pointing to the bottom.

4. Considering all the possible hive configurations, depending on p, what is the average length of

a) a maximal path?
b) a crossing path?

Give bounds as precise as possible.

5. To avoid getting lost in the hive, the bees give different smells to the crossing paths so that, in each cell, two different maximal paths will always have different smells. A corollary of the four-color theorem implies that the bees need not more that 4 different smells. Can they in fact use fewer smells in the following cases (and if so how many fewer)? Whether the answer is yes or no try, as much as possible, to give proofs that do not rely on the four color theorem.

a) For a given hive configuration, how many different smells, at minimum, do the bees have to use?
b) Is it possible to build a hive that requires only 2 smells?
c) What happens if the bees also give a smell to the non-crossing paths?

6. For a given hive configuration, its partition is the set of all the subsets $\{a,b\}$ such that there is a crossing path between a and b. What are the possible partitions of a hive of size N?

7. Suggest and study further directions of research.

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3. Coin tossing

The Casino Royal proposes a new kind of gambling game. On the table, there are two geometric shapes S and S', where S' is contained in S. A coin is tossed and falls into S at random. This means that the probability distribution of the midpoint of the coin is uniform with respect to the area. The player wins if the coin is completely inside S' and loses otherwise.

In the beginning the coin is a disk with diameter $d < 1$.

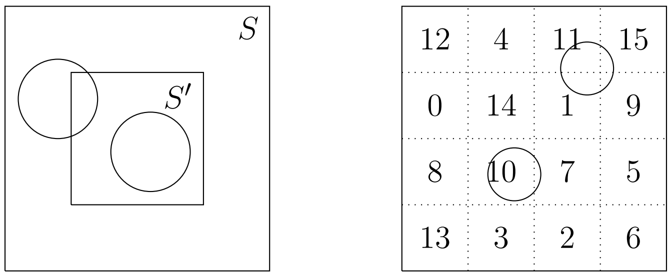

FIGURE 5. Examples of tossed coins. On the left (for Question 1a), we see a coin which wins and another which loses. On the right (for Question 4), one coin wins $11+15+1+9=36$ euro, while the other wins $10+7=17$ euro.

1. In this question, S and S' are regular n-gons (with $n \geqslant 3$) with the same center and orientation, and have sides of lengths 1 and $\ell < 1$ respectively.

a) If $n=4$, determine ℓ (as a function of d) such that the game is fair.
b) Same question for $n=3$.
c) Same question for $n>4$.

The winning condition now changes: the player wins if the coin is entirely inside S' and shows heads, or if it is completely outside S' and shows tails.

2. Is it possible to have a fair game? With the shapes of Question 1, compute the probability of winning as function of ℓ and d.

The croupier proposes the following modification: inside S' there is a third figure S''. The player pays 1 euro and tosses the coin. If the coin falls completely inside S'', the player gets 2 euro back. If the coin falls completely inside S', but not completely inside S'', the player gets 1 euro back. In all other cases, the croupier keeps the money.

3. When S, S' and S'' are regular n-gons with sidelengths $1 > \ell > m$ and the same center and orientation, determine all values of ℓ and m such that on average the player gets 1 euro back. One can start with $n=4$ and $n=3$.

The croupier proposes yet another version. The shape S is now a square with sidelength 1. The shape S is subdivided into n^2 small squares (with $n \geqslant 2$ fixed), which are numbered from 0 to $n^2 - 1$. The player tosses the coin. The player wins s euro, where s is the sum of all numbers associated to the little squares the coin intersects. The right of Figure 5 shows two examples. Naturally, the casino wants to minimize the gain of the player.

4. For $d < \frac{1}{n}$, does the numeration of the small squares influence the average gain of the player? If yes, determine all numerations for which the average gain is minimal and compute this gain. What happens for $d > \frac{1}{n}$?

Instead of taking the sum of all numbers associated with the little squares the coin intersects, the croupier now takes their product.

5. Reconsider Question 4 in this setting.

Now, the coin is not a disk anymore, but a square of sidelength d. When it is tossed at random, the center is still uniformly distributed with respect to area, and its rotation is uniformly distributed in $[0, 2\pi)$.

6. Reconsider all previous questions in this setting.

7. Suggest and study further directions of research.

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4. The rainbow bridge

The rainbow bridge linking the kingdoms of the Yggdrasill world tree is broken. To travel between Ásgarðr, Miðgarðr and Jotunheimr, it is now necessary to wear a necklace that carries a number x belonging to the interval $[0, 1]$. We say that the **position** of the traveller is the kingdom on which they are, and that the **configuration** is the pair (position, number carried on the necklace). The configuration changes according to the following rule.

When leaving Ásgarðr, the rainbow bridge goes to Miðgarðr, leaving the number x on the necklace unchanged unless $x = 1$. If $x = 1$, the bridge goes to Jotunheimr and turns x into 0.

From Miðgarðr, if $0 < x < 0.5$, the bridge goes to Miðgarðr and transforms the number into $1 - 2x$. If $0.5 \leqslant x \leqslant 1$, then the bridge goes to Ásgarðr and transforms the number into $2 - 2x$. If $x = 0$, then the bridge goes to Jotunheimr and x is unchanged.

When leaving Jotunheimr, the bridge always goes towards Ásgarðr and the number on the necklace becomes $x - x^2$.

A configuration is **stable** if it does not change the position and leaves the number on the necklace unchanged. A journey such that after $n \geqslant 1$ steps, the traveller returns to their point of origin with their original number is called an n-gon. A 3-gon is called a triangle and a 4-gon is called a square.

1. How many stable configurations are there?

2. How many triangles are there? And squares?

3. What are the integers n for which there is an n–gon? How many n–gons are there?

4. Are there journeys that never return to the same configuration?

5. *In this question only*, we assume that after every k steps, the traveller remembers how they liked the kingdom they visited exactly p steps ago, and is instantly transported to this kingdom without changing the number. With this new rule, what are the integers n for which there is an n–gon? How many n–gons are there?

6. Let $p \in [0, 1]$. *In this question only*, we assume that Loki, the God of Deceit wants to upset the traveller. At each step, with probability p, she turns the number x on the necklace into $1 - x$.

a) Depending on the initial configuration, what is the probability that the traveller never leaves their initial position?
b) Depending on the initial configuration, what is the probability that the traveller never goes back to their initial position?
c) Let $n \geqslant 2$. Assume that the traveller starts from a configuration that would give an n-gon if $p = 0$. If $p \in (0,1]$, try to estimate as precisely as possible the probability that the journey is not an n-gon.
d) Try to estimate as precisely as possible the probability that for every $n \geqslant 2$ and for every initial configuration, the journey is not an n-gon.

7. Suppose we write on the necklace a random rational number $x \in [0,1)$ and randomly choose a starting location. What is the probability of eventually returning to the initial configuration?

8. Suggest and study further directions of research.

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5. Arithmetic and shopping

In Wonderland, the economists have designed a very innovative monetary system. People can pay only with coins. The value of a coin must be a power of a prime number, and there is a coin for each power of a prime number, *i.e.* there are coins which carry $2, 2^2, 2^3, \ldots, 3, 3^2, \ldots, p, p^2, \ldots$ (for each prime number p). Lewis, who is at the head of a very well-known shop called Tea's shop, wants the clients to have fun while shopping.

1. First, Lewis is interested in products whose prices are powers of 2. How many objects must Lewis have in stock, at the least, to ensure that the clients can always choose two different objects so that the product of their prices is a perfect square?

2. Let $n \in \mathbb{N}$ be fixed. Lewis wonders how many coins that carry a power of 2 are needed to ensure that it is always possible to choose n coins so that the product of their values, x, can be decomposed in the form $x = k^n$ for some k?

3. Lewis is now interested in prices which have only prime divisors smaller than 25. How many objects must Lewis have in stock to ensure that the clients can always choose 4 objects such that the product of their prices is a power of 4?

4. Now, Lewis is interested in sums of prices. How many objects must Lewis have in stock, at least, to ensure that the clients can always choose n objects such that the sum of their prices is divisible by n?

5. Lewis is now interested in prices which can be written in the form $2^a 3^b 5^c$ for some nonnegative integers a, b, c. How many objects must Lewis have in stock, at least, to ensure that whatever their prices are, the clients can always choose n objects so that the product of their prices, p, can be decomposed in the form $p = k^n$ for some $k, n \in \mathbb{N}$ with $n > 1$?

6. Suggest and study further directions of research.

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6. A fence for the goats

Lucie is a goat breeder. She needs to buy a new fence, and would like to minimise the price of the new fence. A **fence** F is a polygon of $\mathbb{R}^2$ composed of a finite (integer) number $n \geqslant 3$ of straight sides. We denote by $O_1, \ldots, O_n \in \mathbb{R}^2$ the consecutive vertices of the polygon F.

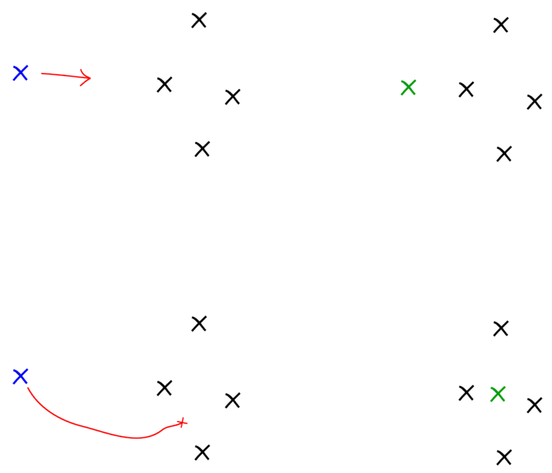

FIGURE 6. Above: the price decreases when the blue cross is replaced by the green one; Below: we do not know (blue cross cannot go continuously to green without, at some point, getting farther from one of the black crosses).

A first fence provider determines the price of a fence in the following way. Let $\ell_1 = d(O_1, O_2)$, ..., $\ell_n = d(O_n, O_1)$ be the lengths of the side of the fences (where $d(O_i, O_j)$ is the Euclidean distance in $\mathbb{R}^2$ between the points O_i and O_j). The **price** of the fence is given by

$$c_1^f(F) = f(\ell_1) + \ldots + f(\ell_n),$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function such that $f(0) = 0$. When f is unambiguous, we may for simplicity replace c_1^f by c_1.

1. Given two strictly increasing functions f and g satisfying $f(0) = g(0) = 0$,

a) is it possible to have two fences F and F' such that $c_1^f(F') < c_1^g(F)$ but $c_1^f(F) > c_1^g(F')$?
b) is it possible to have two fences F and F' such that $c_1^f(F) < c_1^f(F')$ but $c_1^g(F) > c_1^g(F')$?
c) find a criterion (as general as possible) to ensure that for any fence F, $c_1^f(F) \leqslant c_1^g(F)$.

2. In this question, we assume that the number of sides is limited, so Lucie needs to minimise the price of the fence satisfying $n \leqslant n_{\max}$, for some integer $n_{\max} \geqslant 3$. Lucie needs a fence enclosing an area at least A, for a given real number $A > 0$. In terms of A,

a) what are the optimal polygon and the optimal price for $f(x) = x$? Study the limit as $n_{\max} \to +\infty$.
b) what are the optimal polygon and the optimal price for $f(x) = x^2$? Study the limit as $n_{\max} \to +\infty$.
c) give properties as general as possible for arbitrary differentiable f in the limit as $n_{\max} \to +\infty$.

Now Lucie considers another fence provider. We denote their price by $c_2(F)$ or $c_2(O_1, \ldots, O_n)$. The price c_2 is a nonnegative real number, is invariant under translation and rotation of the fence F. Moreover, the price decreases upon a transformation satisfying the following two conditions: (i) a side of the fence is made continuously shorter, and (ii) during this operation (including intermediate steps), for each $i, j \in \{1, \ldots, n\}$, the distance between O_i and O_j never increases. See Figure 6. We do not know by how much this operation decreases the price; in particular, this may also depend on the lengths of all the other fences.

3. Lucie wonders about the differences between the two providers. Assume that there are at most $n_{\max}$ sides on the fence.

a) According to the value of $n_{\max}$, are there examples of prices c_2 which cannot be written as $c_2 = c_1^f$ for any function f?

b) Are there examples of prices c_2 and two sets of points $O_1,\dots,O_n$ and $O'_1,\dots,O'_n$ such that $d(O_i,O_j) \leqslant d(O'_i,O'_j)$ for all $i,j = \{1,\dots,n\}$ but $c_2(O_1,\dots,O_n) > c_2(O'_1,\dots,O'_n)$?

4. Lucie wants to buy a fence with a fixed number of sides from the second provider. Assume that each side has length at least D. How to make the total price as small as possible with respect to the chosen number of sides?

5. Now Lucie has a limited budget: she can spend at most $C > 0$; still, she wants to maximise the area available for her goats.

a) Given a price c_2, does there necessarily exist a fence which maximises the area among the fences with price at most C? (That is, such that no other fence with price at most C encloses a strictly greater area).
b) Give conditions on the price c_2, as loose as possible, to ensure that for all $C > 0$, there exists a fence which maximises the area among the fences with price at most C.

6. Lucie's neighbour breeds birds, which live in a 3-dimensional environment. The fence is now a polyhedron of $\mathbb{R}^3$ with faces which are now triangles. Again, we label the points with $O_1,\dots,O_n \in \mathbb{R}^3$.

a) Assume that the price is determined by the lengths of the edges of the polyhedron (i.e. in all the above, we replace "length of a side" by "length of an edge"). Address questions 2 and 4 in this new situation.
b) Assume that the price is determined by the areas of the triangles (i.e. in all the above, we replace "length of a side" by "area of a face"). Address questions 2 and 4 in this new situation.

7. Suggest and study further directions of research.

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7. GENERALIZED TIC-TAC-TOE

Olivia and Xavier like to play tic-tac-toe. Rapidly bored by the classical version of the game, they invent variations. On an $m \times n$-grid, Olivia marks boxes with a round and Xavier marks boxes with a cross. In their turn, a player has to mark exactly one box which has not been marked before. They play alternatively and Olivia starts.

At the beginning, the goal for a player is to achieve with their symbols some prescribed figure (up to rotation or reflection), called the **winning figure**.

FIGURE 7. Triomino with an angle and L-shaped tetromino

1. Analyze winning strategies when the winning figure is

a) an L-shape tetromino (see Figure 7 on the right).
b) a 2×2-box.

2. Is there a winning figure for which Xavier has a winning strategy?

Olivia proposes to have two winning figures, which are two rectangles $a_1 \times b_1$ and $a_2 \times b_2$. A player wins if their symbols form one of the two rectangles.

3. Analyze winning strategies when

a) a_1, b_1, a_2 and b_2 are strictly bigger than 1.
b) the rectangles are 2×3 and 1×5.
c) the rectangles are 2×2 and $1 \times n$ for some given $n > 4$.

Xavier is unsatisfied with the game. He proposes to reverse the situation: some given figure (up to rotation and reflection) is now forbidden. The player who can not make a move anymore loses.

4. Analyze winning strategies if the losing figure is a 1×2-domino. One can start by treating the case where mn is even.

5. Is there a forbidden figure for which Olivia has a winning strategy for which the game always ends before the entire board is filled?

Xavier now proposes to mix both game versions: a first figure T_1 is winning, while another figure T_2 is forbidden. A player who cannot play anymore loses. A marking is not allowed when it creates a winning and a forbidden figure at the same time.

6. Analyze winning strategies if T_1 is a 1×4-rectangle and T_2 is the triomino with an angle (see Figure 7 on the left).

7. Is there a choice for T_1 and T_2 such that for infinitely many grid sizes, there is a winning strategy for Olivia, and for infinitely many grid sizes there is a winning strategy for Xavier?

Olivia suggests that a player must always mark a box which is adjacent (through a common edge) to at least one other box which is already marked (apart from the very first move).

8. Reconsider the previous questions with this extra constraint.

9. Suggest and study further directions of research.

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8. Polyhedral construction

Sebastian and Susanne like to construct polyhedra. A **polyhedron** is a bounded set of $\mathbb{R}^3$, obtained as the convex hull of a finite set of points. It consists of vertices, edges and faces. We consider two polyhedra as being equal if there is a bijection between the vertices inducing a bijection of edges and faces.

For reasons of aesthetics, Sebastian and Susanne want that every vertex is contained in the same number of edges $r \geqslant 3$; we say that the polyhedron is **r-adjacent**. Initially, they only want to use triangles (not necessarily regular ones).

1. Which polyhedra can they build, depending on r?

Sebastian proposes to analyse 3-adjacent polyhedra more in detail. In addition to their triangles, they now also allow p-gons (again not necessarily regular), with $p \geqslant 4$ fixed.

2. If the faces of a 3-adjacent polyhedron are exactly a triangles and b polygons with p vertices, find a relation between a and b.

3. Are there finitely many 3-adjacent polyhedra whose faces are only triangles and p-gons? If yes, describe all these polyhedra. One can distinguish the following cases:

a) $p = 4$ and $p = 5$.
b) $6 \leqslant p \leqslant 10$.
c) What happens for $p \geqslant 11$?

Susanne prefers r-adjacent polyhedra where $r > 3$.

4. Reconsider Questions 2 and 3 in the case of 4- and 5-adjacent polyhedra.

Sebastian likes symmetric polyhedra, in particular polyhedra P which are centrally symmetric (meaning that if $z \in P$, then $-z \in P$).

5. In the setting of Questions 3 and 4, find all centrally symmetric polyhedra they can build.

6. Suggest and study further directions of research, for example generalized polyhedra which discretize surfaces with g holes.

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9. Landing a probe

A team of engineers is responsible for landing a space probe on Venus. The probe is equipped with a parachute but, due to strong atmospheric currents, the team wants the parachute to detach from the probe before the probe reaches the ground. This will prevent the parachute from being carried away by air currents and will not obstruct the movement of the probe on the ground. The probe will therefore be in free fall from a certain height but has limited solidity. To ensure that the parachute can fly as far as possible, the engineers want the parachute to detach from the probe as high as possible, without the probe subsequently crashing.

We denote by H the maximal height from which the probe will not be destroyed if it is dropped from this height. For simplicity, we assume that H is an integer.

The engineers test the probe as follows: they successively choose heights h_i and see if the probe crashes or remains intact by dropping it from the height h_i (which is also an integer). The engineers only know that $H \in [\![1, K]\!]$ for some fixed $K \in \mathbb{N}$.

1. In this question, we assume that $K = 100$.

a) If they are lucky, how many trials do engineers need at least to be sure of the value of H?
b) Propose a strategy for the engineers to find the value of H, whatever H is.
c) The engineers have established an optimal strategy: they are certain that they only need N trials to find the value of H. Find the value of N in terms of K.
d) Are there several different strategies which satisfy question c)?
e) For each of the optimal strategies, what are the values of H which require exactly N trials to ensure the value of H?
f) For each of the optimal strategies, what are the values of H that can be found in exactly 2 trials? 3 trials?

2. Do Question 1 again for any integer $K > 1$.

3. Now, the engineers want to destroy as few probes as possible during the experiments. How many trials do they need to be sure of the value of H if they want to destroy at most d probes during the tests

a) for $d = 1$?
b) for $d = 2$?
c) for general $d \in [\![1, K]\!]$?

From now on, the engineers want to reuse the probes which have been launched but not broken. A trial which did not break a probe damaged it: if a probe has already been dropped from the heights $h_1, ..., h_k$ without breaking, it can still be launched from any height strictly lower than $H - h_1 - ... - h_k$ without breaking. If the probe is launched from a height $H - h_1 - \ldots - h_k$ or more, the probe will break.

4. The engineers have established an optimal strategy: they are certain that they only need N trials to find the value of H.

a) Estimate N as a function of K.
b) What are the values of H for which N trials are necessary to be sure of the value of H?
c) For an integer $1 < d < N$, what are the values of H for which it is necessary to break d probes to be sure of the value of H by following an optimal strategy?

5. Reconsider Question 4 again assuming that the engineers want to destroy at most d probes in the test phase.

6. Assume now that $n \in [\![1, K]\!]$ probes can be dropped during one single trial. The engineers want to know the value of H in as few trials as possible. They are allowed to break at most d probes. Do questions 3 and 4 again for $n \geqslant 2$.

7. Suggest and study further directions of research.

* * *

10. CATCHING THE RABBIT

Alice and the rabbit play the following game on a graph. A **graph** is a pair (V, E), where V is the set of vertices and $E \subset V \times V$ is a set of edges (such that if $(a,b) \in E$, then $(b,a) \in E$). We say that a **path of length** n is a sequence of edges $(e_1, \ldots, e_n)$ of E which joins a sequence of vertices, we do not assume that a path cannot go through the same vertex twice. This yields a distance $d(v_1, v_2)$ between two vertices v_1, v_2 of G as the minimal length of a path between them. A graph G is said to be **connected** if for all vertices v_1 and v_2 in G we can find a path that connects them. In this problem all graphs are assumed to be connected. We also define two particular kind of graphs:

- The graph $\mathcal{N}$ whose vertices are indexed by the natural integers $\{1, 2, \ldots\}$ and whose edges connect the vertex n with the vertex $n+1$;
- The graph $\mathcal{Z}^2$ whose vertices are indexed by the pairs of integers $\{(i,j) \mid i,j \in \mathbb{Z}\}$ and whose edges connect the vertex (i,j) with the vertex $(i, j+1)$ and with the vertex $(i+1, j)$.

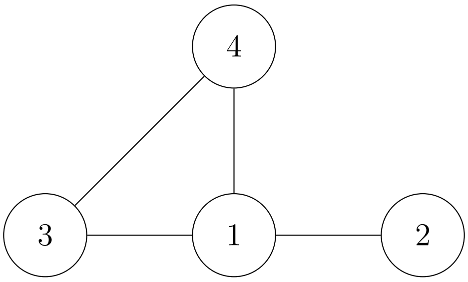

FIGURE 8. An example of graph with 4 vertices and 4 edges.

On a graph G, Alice and a rabbit play a game. The goal of the game for Alice is to catch the rabbit in which case we say that she won. At the start of the game, Alice first places herself on any vertex, then the rabbit places himself on the vertex of his choice (potentially depending on where Alice is). Alice and the rabbit move in turn, and the rabbit starts. Alice and the rabbit move along a path on the graph, always of fixed length $a \in \mathbb{N}$ for Alice and of fixed length $r \in \mathbb{N}$ for the rabbit. They go on as long as Alice does not catch the rabbit.
We way that Alice **catches** the rabbit if she passes it as she moves (or stops at the rabbit's vertex). If it is the rabbit that crosses Alice we do not say that Alice caught him except if he stops at Alice's vertex.

To begin with, we assume that Alice knows the rabbit's position at all times. Similarly, the rabbit knows the location of Alice at each time. We also assume for now that $a = r = 1$.

1. Let k be an integer greater or equal to one.

a) If the graph G is as in Figure 9, is there a strategy that ensures that Alice always catches the rabbit ?
b) Same question if G is a graph as in Figure 10.

2. We assume now that Alice and the rabbit are moving on a graph that can be represented by a square of size $n \times n$ as in Figure 11.

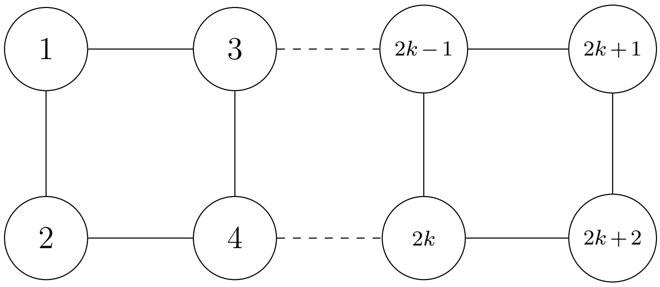

FIGURE 9

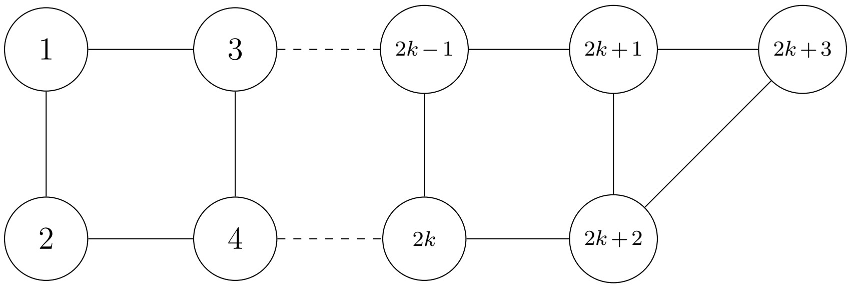

FIGURE 10

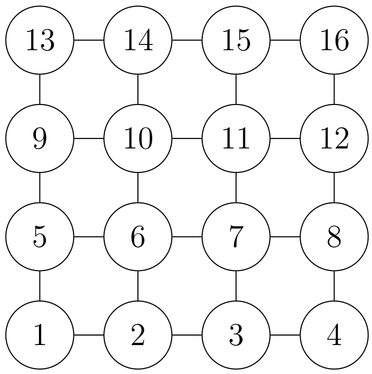

FIGURE 11. A square of size 4×4

a) What is the minimum number of edges that must be removed to ensure that Alice catches the rabbit? One can start by looking at the cases $n = 2, 3, 4$.
b) What is the minimum number of edges that must be added to ensure that Alice catches the rabbit? One can start by looking at the cases $n = 2, 3, 4$.

3. Now, we suppose that the graph is an infinite tree (see Figure 12); it means that there exists a main node and then that each node has n daughters for some fixed $n \geqslant 2$. We assume that the daughters are ordered meaning that there is a first one, a second one etc... In this question, the rabbit has to choose an integer $p \geqslant 1$ and numbers $(m_1, \ldots, m_p) \in [\![1, n]\!]^p$. After that, he has to move according to the periodic pattern $(m_1, \ldots, m_p)$, meaning that he has first to go to the m_1-th daughter, the second time to the m_2-th daughter of the node he is now on, up until the m_p-th daughter, and then continues starting again with the m_1-th daughter of the node he lies in etc. We also assume that Alice does not know the location of the rabbit. But, she has a GPS that gives her a disk of radius R in which she knows the rabbit lies.

a) Assume that Alice knows the value of p. Can she recover the sequence $(m_1, \ldots, m_p)$?
b) Now Alice does not know the value of p but during her turn, she can use a pokeball to choose any vertex and catch the rabbit if he is on it. Can she elaborate a strategy to catch the rabbit?

We do not assume anymore that a and r are equal to 1.

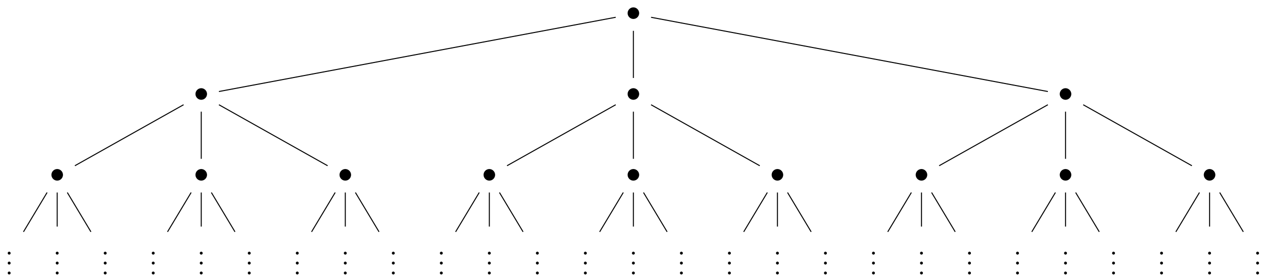

FIGURE 12. An infinite tree, with $n = 3$

4. Now, Alice and the rabbit are moving on a graph with an infinite branch, that is a graph G that can be written as the union of a *finite* graph G' and the graph $\mathcal{N}$ by identifying a vertex $g \in G'$ with the vertex 1 in the graph $\mathcal{N}$. This time the rabbit and Alice start the game at the same vertex v, the vertex v is chosen by Alice.

a) For a given r, does there always exists an a such that Alice can always catch the rabbit for sure?
b) Is it possible that this number a is strictly smaller than r? If yes, find (or characterize) all the graphs that have this property.
c) Is it possible that this number a is equal smaller than r? If yes, find (or characterize) all the graphs that have this property.
d) Find bounds as precise as possible on the minimal a such that Alice can catch the rabbit in terms of G and r.

5. Alice lost the GPS but she bought a sonic radar that can give the distance to the rabbit at the beginning of her turn. They play on $\mathcal{Z}^2$. What condition on a and r ensures that she can stay at bounded distance from the rabbit? What if she wants to catch him?

6. Suggest and study further directions of research.