Syllabus

January

The present document outlines the mathematical concepts that may be necessary for participating in ETEAM.

Unless explicitly stated otherwise, the results listed below are expected to be known by middle school and high school students.

This list is not intended to be an exhaustive enumeration; it highlights guidelines for the "most common cases" and is deliberately ambiguous about certain topics.

Contents

1 Arithmetic

1.1 Numbers and Operations

- Types of numbers: natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers.
- Properties of operations: commutative, associative, distributive.
- Prime numbers, composite numbers, and prime factorization.
- Greatest common divisor (GCD) and least common multiple (LCM).
- Modular arithmetic and congruences.
- Absolute value and its properties.

1.2 Exponents and Roots

- Laws of exponents: $a^m \cdot a^n = a^{m+n}$, $\frac{a^m}{a^n} = a^{m-n}$, $(a^m)^n = a^{mn}$.
- Negative and fractional exponents.
- Square roots, cube roots, and nth roots.
- Simplifying expressions involving exponents and roots.

2 Algebra

2.1 Polynomials

- Definition, degree, and standard form of a polynomial.
- Roots of polynomials: Factor theorem.
- Solving quadratic equations (factoring, completing the square, quadratic formula).
- Relationship between roots and coefficients (Vieta's formulas).

2.2 Functions

- Definition of a function and basic notions (domain, range, injectivity, surjectivity, bijectivity).
- Linear, quadratic, and polynomial functions.
- Exponential and logarithmic functions (basic properties and solving equations).
- Composition of functions and inverse functions.

2.3 Systems of Equations and Inequalities

- Solving linear systems of two or three equations.
- Solving quadratic and rational inequalities.
- Linear programming basics

2.4 Matrices and Determinants

- Definition of matrices, matrix addition, scalar multiplication, and matrix multiplication.
- Determinant of a matrix (2x2 and 3x3).
- Properties of determinants (linearity, cofactor expansion).
- Inverse of a matrix (using the adjoint and determinant).
- Solving systems of equations using matrices (Cramer's Rule and Gauss-Jordan elimination).

3 Geometry

3.1 Plane Geometry

- Properties of triangles, circles, quadrilaterals, and polygons.
- Congruence and similarity of triangles.
- Pythagoras' theorem and its converse.
- Circle theorems (angles, tangents, chords).

3.2 Coordinate Geometry

- Equation of a line: slope-intercept form, point-slope form.
- Distance between two points and midpoint of a segment.
- Equation of a circle.
- Intersection of lines and curves.
- Parametric curves and curves defined by an equation.

3.3 Trigonometry

- Trigonometric ratios (sine, cosine, tangent) and solving right triangles.
- Unit circle and definitions of trigonometric functions for any angle.
- Fundamental identities: $\sin^2(x) + \cos^2(x) = 1$.
- Law of Sines and Law of Cosines.

3.4 Vectors

- Definition and representation in 2D and 3D.
- Dot product and cross product.
- Applications to geometry: equations of lines and planes.

4 Calculus

3.1. Limits and Continuity (Introductory Level)

- Definition of a limit.
- Basic limit properties and computation.
- Continuity at a point and over an interval.

4.1 Sequences and Series

- **Definition of a Sequence:** A sequence is an ordered list of numbers where each term is determined by a specific rule or formula.
- **Definition of a subsequence.**
- **Arithmetic Sequences:**
  - General term: $a_n = a_1 + (n-1)d$, where a_1 is the first term and d is the common difference.
  - Sum of the first n terms: $S_n = \frac{n}{2}(a_1 + a_n)$.
- **Geometric Sequences:**
  - General term: $a_n = a_1 \cdot r^{n-1}$, where r is the common ratio.
  - Sum of the first n terms: $S_n = a_1 \frac{1-r^n}{1-r}$, for $r \neq 1$.
  - Sum of an infinite geometric series (when $|r| < 1$): $S = \frac{a_1}{1-r}$.
- **Special Sequences:**
  - Fibonacci sequence: $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.
  - Factorial sequence: $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$, with $0! = 1$.
- **Properties of Sequences:**
  - Convergence and divergence of a sequence.
  - Monotonic sequences: increasing and decreasing sequences.
  - Bounded monotonic sequences and their properties.
  - Bolzano Weierstrass theorem.
- **Series:**

  - Definition of a series as the sum of the terms of a sequence.
  - Sigma notation: $\sum_{k=1}^{n} a_k$.
  - Telescoping series and partial sums.

4.2 Derivatives

- Definition of the derivative as a rate of change or slope of the tangent.
- Differentiation rules: power rule, product rule, quotient rule, chain rule.
- Applications of derivatives: finding extrema, solving optimization problems.

4.3 Integration (Introductory Level)

- Concept of the integral as an area under a curve.
- Basic antiderivatives and definite integrals.

5 Probability and Statistics

5.1 Probability

- Basic concepts: probability of an event, complementary events, union, and intersection.
- Conditional probability and independence.
- Counting principles: permutations and combinations.

5.2 Statistics

- Measures of central tendency: mean, median, mode.
- Measures of dispersion: range, variance, standard deviation.
- Representing data: histograms, box plots, scatter plots.

6 Discrete Mathematics

6.1 Graph Theory

- **Basic Definitions:**
  - A graph $G = (V, E)$ consists of a set of vertices V and a set of edges E.
  - Types of graphs: undirected, directed (digraphs), weighted, and unweighted graphs.
  - Degree of a vertex: the number of edges connected to it.
  - Special graphs: complete graphs, bipartite graphs, trees, and cycles.
- **Types of Graphs:**
  - **Simple Graph:** No loops or multiple edges.
  - **Multigraph:** May have multiple edges between vertices.
  - **Connected Graph:** There is a path between every pair of vertices.
  - **Complete Graph:** Every pair of vertices is connected by an edge.
  - **Bipartite Graph:** Vertices can be divided into two sets, with edges only between sets.
  - **Planar Graph:** Can be drawn on a plane without edges crossing.
- **Paths and Cycles:**
  - Path: A sequence of vertices where each pair of consecutive vertices is connected by an edge.
  - Cycle: A closed path with no repeated edges or vertices (except the starting and ending vertex).
  - Eulerian path: A path that uses every edge exactly once.
  - Hamiltonian path: A path that visits every vertex exactly once.
- **Trees:**
  - A tree is a connected graph with no cycles.
  - Properties of trees:
    - A tree with n vertices has $n - 1$ edges.
    - Any two vertices in a tree are connected by a unique path.
  - Applications of trees: spanning trees, binary trees, and decision trees.

- **Graph Representations:**
  - Adjacency matrix: A square matrix where the entry a_{ij} represents the presence (or weight) of an edge between vertices i and j.
  - Adjacency list: A list of vertices where each vertex has a list of its neighbors.
  - Edge list: A list of all edges, each represented by its two endpoints (and possibly weights).

7 Complex Numbers

- Definition and representation of complex numbers.
- Operations: addition, subtraction, multiplication, division.
- Modulus, argument, and polar form.
- De Moivre's theorem and applications.

8 Mathematical Reasoning

Warning ! We recall that a mathematical sentence uses words. A proof can't be just a sequence of mathematical symbols.

8.1 Logic and Propositions

- **Statements and Propositions:** A proposition is a declarative statement that is either true or false.
- **Logical Connectives:**
  - Conjunction ($\wedge$): $P \wedge Q$ is true if both P and Q are true.
  - Disjunction ($\vee$): $P \vee Q$ is true if at least one of P or Q is true.
  - Negation ($\neg$): $\neg P$ is true if P is false.
  - Implication ($\Rightarrow$): $P \Rightarrow Q$ is false only when P is true and Q is false.
  - Equivalence ($\Leftrightarrow$): $P \Leftrightarrow Q$ is true when P and Q have the same truth value.
- Sufficient conditions, necessary conditions.

- **Truth Tables:** Constructing truth tables to determine the validity of logical statements.
- **Quantifiers:**
  - Universal quantifier ($\forall$): "For all" or "every."
  - Existential quantifier ($\exists$): "There exists" or "at least one."

8.2 Set theory

Basic Concepts of Sets

- Definition of a set and examples (finite, infinite, empty set)
- Notation: Curly braces { }, element symbol ($\in$, $\notin$)
- Methods of describing sets:
  - **Roster form**: $\{1, 2, 3\}$
  - **Set-builder notation**: $\{x \mid x \text{ is a positive even number}\}$

Types of Sets

- Finite and infinite sets
- Subsets and proper subsets ($\subseteq$, $\subset$)
- Universal set and empty set ($\emptyset$)
- Power set ($\mathcal{P}(A)$)

Set Operations

- **Union** ($\cup$): $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- **Intersection** ($\cap$): $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- **Difference** ($\setminus$): $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$
- **Complement** (A^c): Elements not in A (relative to the universal set)

Venn Diagrams

- Representation of set relationships
- Visualizing unions, intersections, and complements

Cartesian Product & Relations

- Cartesian product of two sets: $A \times B = \{(a, b) \mid a \in A, b \in B\}$
- Ordered pairs and their significance

Applications of Set Theory

- Basic counting principles using sets
- Use in probability (sample spaces, events)
- Simple logic statements with sets

8.3 Methods of Proof

- **Direct Proof:** Proving a statement by logical deduction from known facts and definitions.
- **Proof by Contradiction:** Assuming the negation of the statement to show that it leads to a contradiction.
- **Proof by Contrapositive:** Proving the contrapositive of a statement ($P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$).
- **Mathematical Induction:**
  - Basis step: Show the statement is true for the initial value (e.g., $n = 1$).
  - Inductive step: Assume the statement is true for $n = k$ and prove it is true for $n = k + 1$.
- **Counterexamples:** Demonstrating the falsity of a statement by providing a specific example.